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Normal Surface Singularities with C^* Action

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1. Introduction

Let X be an affine algebraic variety (we always work over the complex numbers $\boldsymbol{C}$) with affine coordinate ring A. It is well known that X admits a $\boldsymbol{C}^*$ action if and only if A admits a grading $A = \oplus_k A_k$, where as usual $A_k \cdot A_l \subset A_{k+l}$. If $A_k = 0$ for $k < 0$ and $A_0 = \boldsymbol{C}$, we say, following Orlik-Wagreich, that the $\boldsymbol{C}^*$ action is *good*. In geometric terms this implies that the point $P \in X$ corresponding to the maximal ideal $\bigoplus_{k>0} A_k$ is the only fixed point of the $\boldsymbol{C}^*$ action. P is called the *vertex* of X.

In this paper we study the case where X is also two dimensional and normal. In particular P is the only point of X which can be singular. We say for short that X is a normal $\boldsymbol{C}^*$ surface singularity.

The simplest example of such singularities is a cone over a smooth proper curve C. By definition its affine coordinate ring can be written

$$\bigoplus_{k \geq 0} H^0(C, \mathscr{L}^k),$$

where $\mathscr{L}$ is an ample invertible sheaf of rank 1 on C. We denote this affine variety by $X(C, \mathscr{L})$. Here is another description of $X(C, \mathscr{L})$: let $F(C, \mathscr{L}) \to C$ be the affine line bundle over C with sheaf of sections $\mathscr{L}^{-1}$. Then the zero section of $F(C, \mathscr{L}) \to C$ has negative self-intersection and therefore can be contracted analytically, and actually algebraically. $X(C, \mathscr{L})$ is the result of the contraction.

We will first prove

Theorem 1.1. *Let X be any normal $\boldsymbol{C}^*$ surface singularity. Then there exists a smooth proper curve C', a finite group G of automorphisms of C' and a G-invariant, ample invertible sheaf of* rank 1 *$\mathscr{L}'$ on C' such that*

a) *G acts on $X(C', \mathscr{L}')$, freely except at the vertex.*

b) *X is analytically isomorphic to the quotient of $X(C', \mathscr{L}')$ by G.*

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This theorem strengthens a result announced by Dolgachev [4]: his result is the same as ours except that he does not have a), which is essential, however, for the applications we have in mind. Dolgachev mentions his proof uses singular Seifert fibrations; an elementary proof along these lines can be reconstructed from [8], proof of 3.6.1 and [9], Theorem 4.3. The addition of a) does not present serious difficulties, and presumably could also be obtained by an extension of Dolgachev's method. We present a proof, however, for the sake of completeness, in Section 3. The idea of the proof is to reconstruct X via data given by a certain resolution of X. This resolution is described, following Orlik-Wagreich [8] in Section 2.

In the rest of this paper we draw some conclusions from 1.1 which will also be needed in our study of the deformation theory of X. In Section 4 we construct a natural $\boldsymbol{C}^*$ compactification of X and determine its singularities. In Section 5 we obtain a nice description of the graded pieces A_k of the affine ring of X in terms of some divisors $D^{(k)}$ on the "central curve" E of the resolution of X. The sheaves $\mathcal{O}_E(D^{(k)})$ play for X the same role as $\mathcal{L}^k$ on the cone $X(C, \mathcal{L})$. In particular we compute the genus of X from the cohomology of the $D^{(k)}$. It is important to note that the $D^{(k)}$ can be defined independently of Theorem 1.1. Finally in Section 6 we illustrate the results of Section 5 by computing the equations of X in a few cases.

In a forthcoming paper we apply these results to the deformation theory of X. Note finally that we have restricted to $\boldsymbol{C}$ for two reasons: to avoid difficulties when taking quotients by finite groups, and to apply Fenchel's conjecture (3.1) which is established by transcendental methods.

2. The Minimal Good Resolution of X

A resolution $Z \to X$ of a surface singularity X is *good*, if

1) All the components of the exceptional divisor of $Z \to X$ are smooth and intersect transversally.

2) Not more than 2 components pass through any given point

3) 2 different components intersect at most once.

It is well known (and easy to see) that there is a minimal resolution having these properties.

In this section we describe the minimal good resolution of a normal $\boldsymbol{C}^*$ surface singularity, following Orlik-Wagreich [8].

The *weighted dual graph* of a good resolution is the graph each vertex of which represents a component of the exceptional divisor, weighted by its self-intersection. Two vertices are connected if the corresponding components intersect.

In the minimal good resolution of a normal $\boldsymbol{C}^*$ surface singularity there is at most one component of the exceptional divisor which has positive genus and/or intersects more than two other components. Such a component, if one exists, is called the *central curve* and is denoted E (if there is no central curve then X is a cyclic quotient singularity: see 2.2 below). The dual graph of the minimal good resolution of X is therefore a *star*, that is, a connected tree where at most one vertex is connected to more than 2 other vertices. The connected components of the graph minus the central curve are called the *branches* of the graph and are indexed by i, $1 \leqq i \leqq n$. The curves of the i-th branch are denoted by E_{ij}, $1 \leqq j \leqq r_i$, where E_{i1}

intersects E and E_{ij} intersects $E_{i,j+1}$. Let $b=-E\cdot E$, and $b_{ij}=-E_{ij}\cdot E_{ij}$. Then $b_{ij}\geqq 2$ and $b\geqq 1$.

Finally, set
$$d_i/e_i=b_{i1}-\cfrac{1}{b_{i2}-\cfrac{1}{\ddots\,-\cfrac{1}{b_{ir_i}}}}$$
$$=[b_{i1},\ldots,b_{ir_i}]$$

with $e_i<d_i$, and e_i and d_i relatively prime.

Using the results of [8], it is easy to see we have the following

Theorem 2.1. *Let X be a normal C^* surface singularity. The singularity of X at its vertex is determined up to analytic isomorphism by*

i) *the weighted dual graph of the minimal good resolution.*

If there is no central curve this suffices (see 2.2). Otherwise we need

ii) *the analytic type of the central curve E.*

iii) *The conormal sheaf $\mathscr{L}$ of E in the resolution.*

iv) *The points $P_i=E\cap E_{i1}$ on E.*

We call i)–iv) the *data* of X. Conversely given any set of data as in the theorem, there exists a (unique) normal C^* surface singularity having this data, provided that the intersection matrix given by the graph in i) is negative definite; this condition can be written

$$b-\sum_i e_i/d_i>0. \qquad (*)$$

Throughout this paper $\sum_i$ means sum over i, $1\leqq i\leqq n$.

In case there is no central curve we have the following well known result (Brieskorn [2]):

Lemma 2.2. *Let $\tilde{X}$ be the minimal good resolution of any normal surface singularity X such that all the components of the exceptional divisor are rational and the weighted dual graph is*

Let $d/e=[b_1,\ldots,b_r]$, e and d relatively prime. Then X is analytically isomorphic to the quotient of C^2 by the cyclic group G of order d, acting by $(x,y)\mapsto(\zeta x,\zeta^e y)$, where ζ is a d-th root of unity.

We call this singularity the *cyclic quotient singularity of type* (d,e). Notice that the action of G is free except at the origin of C^2. Therefore the lemma yields a proof of Theorem 1.1 in the case there is no central curve: take $C'=P^1$, $\mathscr{L}'=\mathcal{O}(1)$ and G acting on $\mathcal{O}(1)$ as above. (Actually of course 1.1 is clear for all quotient singularities.)

2.3. *Therefore from now on we restrict to the case there is a central curve.*

Another way of characterizing the singularities of 2.2 is that they are precisely the normal surface singularities with a $\boldsymbol{C}^* \times \boldsymbol{C}^*$ action: [7], p. 35.

3. Proof of the Theorem

We continue with the notation of Sections 1 and 2. To prove Theorem 1.1 we must first construct a cone $X(C', \mathscr{L}')$. The first step is

Fenchel's conjecture 3.1. (Bundgaard-Nielsen [3] and Fox [5]) *Let C be a smooth proper curve over $\boldsymbol{C}$, $P_1, \ldots, P_n$ distinct points of C, and $d_1, \ldots, d_n$ integers greater than 1. Then there exists a smooth proper curve C' and a Galois cover $\pi: C' \to C$ such that*

i) *there is ramification only above the P_i, $1 \leqq i \leqq n$.*

ii) *The ramification index of any point Q_i above P_i ($\pi(Q_i) = P_i$) is d_i.*

Condition ii) means that the stabilizer subgroup of Q_i has order d_i; in other words, if d is the order of the Galois group G of the cover, then there are d/d_i points above P_i. Note that in 3.1 the order of G is *not* specified. We only know that d is a multiple of all the d_i.

We will apply 3.1 with $C = E$, the central curve in the resolution of X, P_i, and d_i keeping the same meaning as in Section 2. To avoid double indices we just write Q_i for a point above P_i. $\sum_{Q_i}$ will always mean sum over all $Q \in C'$ such that $\pi(Q) = P_i$.

3.2. We apply the Hurwitz formula to $\pi: C' \to C$. If K is a canonical divisor on C, then

$$K' = \pi^{-1}(K) + \sum_i \sum_{Q_i} (d_i - 1) Q_i \tag{1}$$

is a canonical divisor on C'. Thus if g (resp. g') is the genus of C (resp. C'),

$$2g' - 2 = d(2g-2) + \sum_o d(d_i - 1)/d_i \tag{2}$$

Any divisor on C' invariant under G can be written

$$\pi^{-1}(D) - \sum_i \sum_{Q_i} f_i Q_i \quad \text{with} \quad 0 \leqq f_i < d_i, \tag{3}$$

for some divisor D on C. Such a divisor is of positive degree when

$$\operatorname{degree} D > \sum_i f_i/d_i. \tag{4}$$

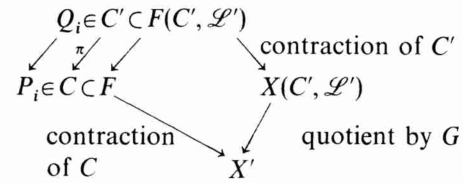
3.3. To any divisor D on C we associate an invertible sheaf of rank 1, $\mathcal{O}_C(D)$, in the usual way (cf. Serre [12], p. 20). Conversely any invertible sheaf of rank 1 can be written in this way. For later on it is useful to recall that $H^0(C, \mathcal{O}_C(D))$ can be identified with the meromorphic functions f on C such that $(f) \geqq -D$, where (f) is the principal divisor associated to f. Note that the degree of $\mathcal{O}_C(D)$ is just the degree of D.

3.4. We now apply 3.1 to $C=E$, etc. Recall that $\mathscr{L}$ is the conormal sheaf of E, the central curve of the resolution of X. Write $\mathscr{L}=\mathcal{O}_E(D)$ for some divisor D on E. The degree of D is b. Let D' be the divisor on C' given by

$$D'=\pi^{-1}(D)-\sum_i \sum_{Q_i} e_i Q_i,$$

where e_i is as in Section 2. Finally set $\mathscr{L}'=\mathcal{O}_{C'}(D')$. Since D' is invariant under G, G acts on $\mathscr{L}'$ and hence on $F(C',\mathscr{L}')$ in a natural manner.

We will show that C', G, and $\mathscr{L}'$ as above satisfy Theorem 1.1. First note that by (*) of Section 2 and formula (4) of 3.2, $\mathscr{L}'$ has positive degree and is therefore ample. Thus we can contract the zero section of $F(C',\mathscr{L}')$, which we call C' by abuse of language, thus obtaining $X(C',\mathscr{L}')$. Let F be the quotient of $F(C',\mathscr{L}')$ by G. The image C in F of the zero section of $F(C',\mathscr{L}')$ is isomorphic to E, and the restriction of the quotient map $F(C',\mathscr{L}')\to F$ to the zero section C' is just $\pi: C'\to C$. The following diagram summarizes the situation.



We must show that the action of G on $X(C',\mathscr{L}')$ is free outside of the vertex and that X' is isomorphic to X.

Lemma 3.5. *G acts freely on $F(C',\mathscr{L}')$ except at the points Q_i. Therefore the quotient F is smooth except at the points P_i. At P_i F has a cyclic quotient singularity of type (d_i, e_i).*

We defer the proof. Note that the lemma implies that G acts freely on $X(C',\mathscr{L}')$ minus the vertex. To show X' is isomorphic to X we use Theorem 2.1: it is sufficient to show that the data of X' is the same as that of X. Therefore we must construct the minimal good resolution of X'. It is not hard to see that resolution is just the minimal resolution $\tilde{F}$ of the cyclic quotient singularities of F. Let $\tilde{C}$ be the proper transform of C in F. By Lemmas 3.5 and 2.2 the data of X' and X will be the same, if we can show

Lemma 3.6. *The conormal bundle of $\tilde{C}$ in $\tilde{F}$ is isomorphic to that of E in the minimal good resolution of X, via the isomorphism $\tilde{C}\to C=E$ induced by $\tilde{F}\to F$.*

Thus the theorem is proved once we have established Lemmas 3.5 and 3.6. Note that the *degree* of the conormal bundle is computed in [9], Theorem 4.3.

Proof of 3.5. We first exhibit the action of G on $F(C',\mathscr{L}')$ explicitly.

Let U_i, $1\leqq i\leqq n$ be a cover of C' by G-invariant Zariski open sets such that $P_i\in\pi(U_i)$, and $P_j\notin\pi(U_i)$, $i\neq j$. Let $h_i\in\Gamma(U_i,\mathcal{O}_{U_i})$ be a local equation for the divisor $\sum_{Q_i} Q_i$. h_i transforms under $g\in G$ by

$$g(h_i)=a_{ig}h_i, \quad \text{for some} \quad a_{ig}\in\Gamma(U_i,\mathcal{O}^*_{U_i}). \tag{*}$$

Since $\sum_{Q_i} d_i Q_i = \pi^{-1}(P_i)$, we can choose h_i so that h_i raised to the power d_i is invariant under G. Therefore a_{ig} is a d_i-th root of unity.

Let $f_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ be transition functions for $\mathcal{L}'$. Then, up to a factor invariant under G,

$$f_{ij} = h_j^{e_j} / h_i^{e_i}.$$

Therefore f_{ij} transforms under $g \in G$ by

$$g(f_{ij}) = a_{ig}^{e_i} f_{ij} a_{jg}^{-e_j}.$$

This shows, as expected, that the class of $\{f_{ij}\}$ in $H^1(C', \mathcal{O}^*)$ is the same as that of $\{g(f_{ij})\}$.

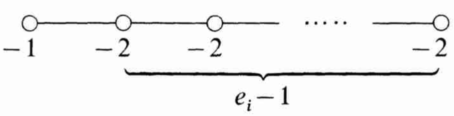
Let t_i be the fibre coordinate of $F(C', \mathcal{L}')$ above U_i. By definition the glueing over $U_i \cap U_j$ is given by $t_i = f_{ij} t_j$. Since the action of G must be compatible with the glueing, we must have

$$g(t_i) = a_{ig}^{e_i} t_i. \qquad (**)$$

We can now prove the lemma. It is clear that the action of G is free except possibly on the fibres above the Q_i, $\pi(Q_i) = P_i$. Pick one such point Q_i. Let G' be the subgroup of G stabilizing it. We need only consider what happens when $g \in G'$. G' is cyclic of order d_i. Let g be a generator of it. Then a_{ig} is a *primitive* d_i-th root of unity, for otherwise the index of ramification of $\pi: C' \to C$ at P_i would be less than d_i. By (**) this shows that the action of G on $F(C', \mathcal{L}')$ is free on the fibre above Q_i for $t_i \neq 0$, thus proving the first assertion of the lemma. On the other hand notice that uniformizing parameters at Q_i are t_i and h_i. By (*) and (**), and since a_{ig} is a primitive d_i-th root of unity for a generator g of G', we see that the action of G gives rise to a cyclic quotient singularity of type (d_i, e_i) at $P_i \in F$. This concludes the proof.

Proof of 3.6.

Lemma 3.7. *Consider the (possibly singular) surface W' obtained as follows: for each i blow up $F(C', \mathcal{L}')$ at Q_i, then blow up again at the intersection of the proper transform of the zero section C' with the first exceptional divisor, then again at the intersection of the proper transform of C' with the new exceptional divisor, and so on, until e_i blow ups have been performed at each Q_i. Call the surface thus obtained $\tilde{W}'$. The weighted dual graph of the exceptional divisor of $\tilde{W}' \to F(C', \mathcal{L}')$ above Q_i is*



and only the -1 curve intersects the proper transform of C'. Blow down all the curves with self-intersection -2, and call the surface obtained W'. Then G acts on W', and the quotient W dominates F and is dominated by $\tilde{F}$ (in other words we have morphism $\tilde{F} \to W \to F$). Furthermore W is smooth along the proper transform C_0 of $C \subset F$.

3.7 is similar to [8], Lemma 3.6.3. Before proving it, let us complete the proof of 3.6, and hence of Theorem 1.1.

Let $\tilde{C}'$ be the proper transform of $C' \subset F(C', \mathcal{L}')$ in $\tilde{W'}$. Since e_i blow ups were performed for each Q_i above P_i and for all i, and since the conormal sheaf of C' in $F(C', \mathcal{L}')$ is by definition

$$\mathcal{L}' = \mathcal{O}_{C'}(\pi^{-1}(D) - \sum_i \sum_{Q_i} e_i Q_i),$$

the conormal sheaf of $\tilde{C}'$ in $\tilde{W}'$ is $\mathcal{O}_{\tilde{C}'}(\pi^{-1}(D))$, with the obvious abuse of language consisting of identifying divisors on $\tilde{C}'$ and C'. By 3.7 the blowing down of $\tilde{W}'$ to W' is an isomorphism in a neighborhood of $\tilde{C}'$, so that the conormal sheaf does not change. Finally, since W is smooth along C_0, the conormal sheaf of C_0 in W is $\mathcal{O}_{C_0}(D)$; since $\tilde{F} \to W$ is an isomorphism along C_0 (by 3.7 again), we are done.

Proof of 3.7. It is clear that the statement of 3.7 is local around each one of the Q_i. One then notices that W is just the first step in Fujiki's resolution of the cyclic quotient singularities X_{de} of type (d, e) [6]. His resolution is the geometric realization of the desingularization of X_{de} by toroidal embeddings [7] p. 35. X_{de} is the quotient of $X = \boldsymbol{C}^2$ by the cyclic group G_d of order d acting on X by $(x_1, x_2) \mapsto (\zeta x_1, \zeta^e x_2)$, ζ a d-th root of unity. Let $Y = \boldsymbol{C}^2$ be a cover of X given by $x_1 = y_1$, $x_2 = y_2^e$. X is the quotient of Y by the cyclic group G_e acting by $(y_1, y_2) \mapsto (y_1, \eta y_2)$, η an e-th root of unity. Let Y' be the blow up of the origin of Y, W' the quotient of Y' by G_e, and W the quotient of W' by G_d. W is called the first step of the resolution of X_{de}. W has at most one singularity, which is a cyclic quotient. Repeating this construction to its singularity, and so on, we eventually reach the *minimal* resolution of X_{de}. Furthermore W' has only a $(e, e-1)$ cyclic quotient singularity. From these facts we deduce 3.7 without difficulty. For more details see [6] or [11].

4. The C^* Compactification of X

X is as always a normal $\boldsymbol{C}^*$ surface singularity. We can compactify X in a natural way using the $\boldsymbol{C}^*$ action by taking $\bar{X} = \operatorname{Proj}(A[t])$, where t is given degree 1 for the grading of $A[t]$. Of course $\bar{X} - X = E_\infty$ is isomorphic to E, the central curve of the resolution of X. If there is no central curve (Lemma 2.2), then $\bar{X}$ can be studied by taking the quotient of $\boldsymbol{P}^2$ by the group G acting as in 2.2. We leave this case to the reader, and now assume there is a central curve.

We study the singularities of $\bar{X}$ along E_∞, by applying 1.1. Let $\bar{F}(C', \mathcal{L}')$ be the projective line bundle associated to $F(C', \mathcal{L}')$ by compactification of the fibres, and $\bar{X}(C', \mathcal{L}')$ the contraction of the zero section of $\bar{F}(C', \mathcal{L}')$. G acts on $\bar{F}(C', \mathcal{L}')$ and $\bar{X}(C', \mathcal{L}')$, and it is easy to see that $\bar{X}$ is just the quotient of $\bar{X}(C', \mathcal{L}')$ by G. Now $\bar{X}(C', \mathcal{L}') - X(C', \mathcal{L}') = C'_\infty$ is isomorphic to C' in an obvious way, and via this isomorphism we obtain points $Q'_i \in C'_\infty$.

Lemma 4.1. *G acts freely on $\bar{X}(C', \mathcal{L}')$ except at the vertex and Q'_i. The quotient $\bar{X}$ has a cyclic quotient singularity of type $(d_i, d_i - e_i)$ at the image P'_i of Q'_i, $1 \leqq i \leqq n$.*

The proof is the same as that of 3.5, noting that local parameters at Q'_i are h_i and $1/t_i$, in the notation of 3.5.

Now let $\tilde{E}_\infty$ be the proper transform of E_∞ in the minimal resolution $\tilde{\bar{X}}$ of the singularities of $\bar{X}$ on E_∞.

Lemma 4.2. *The normal sheaf of $\tilde{E}_\infty$ in $\tilde{\bar{X}}$ is isomorphic to $\mathcal{O}_E(D - \sum_i P_i)$ via the obvious isomorphisms $\tilde{E}_\infty \xrightarrow{\sim} E_\infty \xrightarrow{\sim} E$.*

The proof is the same as that of 3.6.

This gives a complete description of $\bar{X}$, which will be used when we study deformations of X. This should be compared to [10].

5. The Cone-like Structure of X

Recall that the affine ring of X is $A = \bigoplus_{k \geq 0} A_k$. We continue to use the notation of Section 2. For any k let $D^{(k)}$ be the divisor on E:

$$D^{(k)} = kD - \sum_i \{ke_i/d_i\} P_i, \tag{$*$}$$

where D is any divisor such that $\mathcal{O}_E(D)$ is the conormal sheaf $\mathcal{L}$ of E, and for any $a \in \boldsymbol{R}$, $\{a\}$ is the least integer greater than, or equal to a. If a group G acts linearly on a vector space V, denote by V^G the subvector space of invariants.

Theorem 5.1. $A_k = H^0(E, \mathcal{O}_E(D^{(k)}))$.

Proof. Theorem 1.1 shows that $A_k = H^0(C', \mathcal{O}_{C'}(kD'))^G$ where D' is as in 3.4. We have the following easy

Lemma 5.2. *Let $\pi: C' \to C$ be a Galois extension of smooth proper curves with group G, D a divisor on C' invariant under G and $\tilde{D}$ the greatest divisor of C such that $\pi^{-1}(\tilde{D}) \leqq D$. Then*

$$H^0(C, \mathcal{O}_C(\tilde{D})) \xrightarrow{\sim} H^0(C', \mathcal{O}_{C'}(D))^G,$$

where the isomorphism is the pullback of meromorphic functions (cf. 3.3).

To prove Theorem 5.1, just notice that $D^{(k)}$ is the greatest divisor on E such that $\pi^{-1}(D^{(k)}) \leqq kD'$. To see this recall that $\pi^{-1}(P_i) = d_i \sum_{Q_i} Q_i$. Thus 5.2 proves 5.1.

Conversely, given a smooth proper curve E, a divisor D on E, points P_i, $1 \leqq i \leqq n$, and pairs of integers (d_i, e_i), $e_i < d_i$ and e_i and d_i relatively prime, we can form the divisor $D^{(k)}$ as in ($*$). Since $D^{(k)} + D^{(l)} \leqq D^{(k+l)}$ (because $\{a\} + \{b\} \geqq \{a+b\}$),

$$\bigoplus_{k \geqq 0} H^0(E, \mathcal{O}_E(D^{(k)}))$$

is a graded algebra. If degree $D > \sum_i e_i/d_i$, this is the graded ring of the normal $\boldsymbol{C}^*$ surface singularity with the appropriate data. Notice that the construction does not involve the group G. This provides a convenient method for obtaining the affine ring of such singularities from the data of their minimal good resolution. Examples are worked out in the next section.

Definition 5.3. The genus of a surface singularity X is the dimension of $R^1 f_* \mathcal{O}_Z$, where $f: Z \to X$ is any resolution of X [1].

If the genus is 0, X is said to be *rational*. If the genus is 1, X is *strongly elliptic*. If X is affine, then $R^1 f_* \mathcal{O}_Z = H^1(Z, \mathcal{O}_Z)$ by the Leray spectral sequence.

Let X be once again a normal C^* surface singularity, $f: \tilde{F} \to X$ the resolution constructed in Section 3. f factors through

$$\tilde{F} \xrightarrow{g} F \xrightarrow{h} X.$$

Lemma 5.4. *$R^1 g_* \mathcal{O}_{\tilde{F}} = 0$, so that $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}) = H^1(F, \mathcal{O}_F)$.*

The first statement follows because the singularities of F, being cyclic quotient singularities, are rational [2]. The second is an immediate consequence of the Leray spectral sequence.

Lemma 5.5. $H^1(F, \mathcal{O}_F) = H^1(F(C', \mathcal{L}'), \mathcal{O}_{F(C', \mathcal{L}')})^G = \bigoplus_{k \geqq 0} H^1(C', \mathcal{L}'^k)^G$.

The proof is immediate from the definitions.

Lemma 5.6. $H^1(C', \mathcal{L}'^k)^G = H^1(E, \mathcal{O}_E(D^{(k)}))$.

Since $\mathcal{L}' = \mathcal{O}_{C'}(D')$, $H^1(C', \mathcal{L}'^k) = H^0(C', \mathcal{O}_{C'}(K' - kD'))$, where K' is a canonical divisor on C'. Write out K' according to the Hurwitz formula [3.2, Formula (1)] and D' according to 3.4. Finally apply 5.2, noting that $D^{(k)} = kD - \sum_i [(ke_i + d_i - 1)/d_i] P_i$, where for $a \in \mathbf{R}$, $[a]$ is the greatest integer less than, or equal to, a.

Theorem 5.7. *The genus of X is $\sum_{k \geqq 0} \dim H^1(E, \mathcal{O}_E(D^{(k)}))$.*

This follows immediately from the 3 preceding lemmas.

Corollary 5.8. *Let X be a normal C^* surface singularity, with data as in 2.1. Then X is rational if and only if either*

i) *there is no central curve, or*

ii) *the central curve is rational and $kb - \sum_i \{ke_i/d_i\} > -2$, for all $k > 0$.*

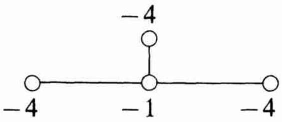
By the same method one can compute the genus of X from b, n, and the e_i and d_i when the central curve is rational.

One should notice the striking resemblence Theorems 5.1 and 5.7 present with the case of cones.

6. Examples

We compute the equations of certain normal C^* surface singularities in order to illustrate Theorem 5.1. In [11] we computed the equations of all the rational double points by this method.

Example 6.1. The central curve is rational and the weighted dual graph is:



Let D be P_3. Then $D^{(k)} = kP_3 - \{k/4\} \sum_i P_i$. Let t be a meromorphic function on E with values $0, 1, \infty$ at P_1, P_2, P_3. Then it is easy to check, using 5.1, that the ring A is generated by an element x of degree 3 and 2 elements y and z of degree 4. As global sections of $H^0(E, \mathcal{O}_E(D^{(k)})$ they can be written:

$$D^{(3)} = 2P_3 - P_1 - P_2, \quad x = t(t-1).$$

$$D^{(4)} = 3P_3 - P_1 - P_2, \quad y = t(t-1), \quad z = t^2(t-1).$$

The relation between x, y and z of minimal degree occurs in degree 12:

$$yz(z-y) = x^4.$$

By 5.7 this singularity is strongly elliptic. It is one of the 14 exceptional singularities of Arnold.

Example 6.2. The central curve is rational, and there are 4 branches with all $e_i = 1$ and all $d_i = 2$. Let $D = 3P_4$. Then $D^{(k)} = 3kP_4 - \{k/2\} \sum_i P_i$. Let t take the values $0, 1, a, \infty$ at $P_1, \ldots, P_4$. Then the graded ring is generated by elements of degree 2 and 3.

$$D^{(2)} = 5P_4 - P_1 - P_2 - P_3, \quad x_1 = t(t-1)(t-a)$$

$$x_2 = t^2(t-1)(t-a)$$

$$x_3 = t^3(t-1)(t-a).$$

$$D^{(3)} = 7P_4 - 2P_1 - 2P_2 - 2P_3, \quad y_1 = t^2(t-1)^2(t-a)^2$$

$$y_2 = t^3(t-1)^2(t-a)^2.$$

It is easy to check that the relations between the x and the y are generated by the 2×2 minors of the matrix

$$\begin{vmatrix} x_1 & x_2 & y_1 \\ x_2 & x_3 & y_2 \\ y_1 & y_2 & (x_3 - x_2)(x_2 - ax_1) \end{vmatrix}.$$

This could also have been computed directly from the equation of the cone $X(C', \mathcal{L}')$, which in this case is a cone over an elliptic curve of degree 2, or also by the results of Wahl (since this singularity is rational).

Example 6.3. The central curve is elliptic of self-intersection -1, and there is one branch with $e = 2$, $d = 3$. The point of intersection of the branch with E is P. $\mathcal{L} = \mathcal{O}_E(R)$ for a point $R \in E$. Thus $D^{(k)} = kR - \{2k/3\} P$. We will distinguish 3 cases (in what follows, $=$ means "linearly equivalent").

i) $R=P$. Then A is generated by 3 elements x, y, z of degree 1, 6 and 9 respectively. The equation between them is

$$z^2=4y^3+g_2yx^6+g_3x^{18},$$

where g_2 and g_3 are constants depending on the modulus of the elliptic curve. The genus of the singularity can be computed by 5.7: it is 3.

ii) $R \neq P$, but $2R=2P$. Then A is again generated by 3 elements x, y, and z, but this time of degree 2, 3, and 7. A typical singularity satisfying these conditions is $z^2=x(y^4+x^6)$. The genus is 2.

iii) $2R \neq 2P$. Then it is easy to check that X is not a hypersurface, and that the genus is 1.

Parts i) and ii) of this example were pointed out by Laufer in a lecture.

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Note added in proof: It is tacitly understood in 3.1 that the case C rational and $n \leqq 2$ has been excluded. That case yields singularities with no central curve, which can be treated directly by 2.2. For the connection between Fuchsian groups and covers of Riemann surfaces see Bers, L.: Uniformization, moduli and Kleinian groups. Bull. London Math. Soc. **4**, 257—300 (1972)

Generalizations of 5.1 to char. p and higher dimension will be dealt with elsewhere.