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Positive Definite Functions on Abelian Semigroups

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Introduction

The moment sequences of Hausdorff and the completely monotone functions on the half-line are closely related concepts and in the theory of capacities of Choquet [2] we find a unified treatment in a study of functions on a semigroup which are monotone of infinite order.

Recently Ressel [5] proved that completely monotone functions on the half-line are characterized among the bounded continuous functions $f:[0,\infty[\to\mathbb{R}$ by the following positive definiteness property:

For every finite set of numbers $s_1,\ldots,s_n\geqq 0$ the matrix $(f(s_i+s_j))_{i,j=1,\ldots,n}$ is positive semi-definite.

This notion makes perfectly sense on any abelian (topological) semigroup, and it is the purpose of the paper to examine this and related notions.

It turns out that for discrete abelian semigroups S the positive definite bounded functions f, normalized such that $f(0)=1$, form a Choquet simplex and the extreme points are exactly the real characters on S, i.e. the multiplicative functions $\varrho:S\to[-1,1]$ such that $\varrho(0)=1$. This implies that the cone of positive definite and bounded functions contains the cone of monotone functions of order infinity, and the two classes coincide in the case of a $2-$ divisible semigroup S i.e. a semigroup where every element is of the form 2a for some $a\in S$.

In analogy with the group case we also study negative definite functions, which by definition are non-negative functions f on S for which $(f(s_i)+f(s_j)-f(s_i+s_j))_{i,j=1,\ldots,n}$ is positive semi-definite for every finite set of elements $s_1,\ldots,s_n\in S$.

These functions can be characterized by a "Schoenberg-theorem" and an integral representation analogous to the Lévy-Khinchin formula.

The functions which are alternating of order infinity form a subcone of the cone of negative definite functions and again the two classes coincide in the 2-divisible case.

For a discrete abelian semigroup S the set of characters $\hat{S}$ forms a compact semigroup in the topology of pointwise convergence and it looks temptating to try to make a duality theory and consider topological semigroups. In analogy with the group case one could hope that an integral representation of continuous positive definite functions on for example a compact semigroup S could be found by considering $\hat{S}_d$, where S_d is S with the discrete topology, and then proving that the representing measure was supported by the continuous characters. This is not true however and several examples show that there is little hope to make a good theory for topological semigroups.

For the classical semigroup $[0, \infty[$ with addition there is however only one non-continuous character and there the discrete representation theorems obtained imply the classical representation theorems for completely monotone functions and for Bernstein functions.

Long ago Hilbert proved that in dimensions $n \geqq 3$ there exist real polynomials P such that $P(x) \geqq 0$ for all $x \in \mathbb{R}^n$ but which cannot be written as a finite sum of squares of polynomials. By means of the integral representation for positive definite functions on the semigroup $\mathbb{N}_0^n$ we prove that there always exists a sequence of polynomials P_k, $k=1,2,\ldots$ approximating P in the norm $\sum |c_\alpha|$, sum of the absolute values of the coefficients, and such that each P_k is a sum of squares of polynomials.

Contents

Chapter I. Positive and Negative Definite Functions on Abelian Semigroups

§ 1. The Associated Banachalgebra $l^1(S)$

1.1. In all of this chapter S denotes an arbitrary *abelian semigroup*, i.e. S is a set equipped with a composition rule called addition and denoted $+$, such that the commutative and the associative law hold. We further assume the existence of a neutral element denoted 0.

1.2. To every abelian semigroup S we associate a commutative real Banach algebra with unit $l^1(S)$. The elements of $l^1(S)$ are the functions $f: S \to \mathbb{R}$ for which

$$\|f\| = \sum_{a \in S} |f(a)| < \infty,$$

and $\|\cdot\|$ is the norm in question. The multiplication in $l^1(S)$ is the "convolution"

$$(f * g)(a) = \sum_{\substack{(s,t) \in S^2 \\ s+t=a}} f(s) g(t).$$

For $a \in S$ we denote by ε_a the element in $l^1(S)$ given by $\varepsilon_a(s)=0$ for $s \neq a$ and $\varepsilon_a(a)=1$.

1.3. Definition. A *character* on S is a function $\varrho: S \to [-1, 1]$ satisfying

(i) $\varrho(0)=1$.
(ii) $\varrho(s+t)=\varrho(s)\varrho(t)$ for all $s, t \in S$.

The set of all characters on S is denoted $\hat{S}$, which is an abelian semigroup under pointwise multiplication, and the neutral element is the character $s \to 1$. We shall equip $\hat{S}$ with the topology of pointwise convergence, which turns $\hat{S}$ into a compact Hausdorff space.

The following result is easily established:

1.4. Proposition. *For every $\varrho \in \hat{S}$ the mapping $\delta_\varrho: l^1(S) \to \mathbb{R}$ given by*

$$\delta_\varrho(f)=\sum_{s \in S} f(s)\varrho(s)$$

is a character on the commutative Banachalgebra $l^1(S)$ and the mapping $\varrho \to \delta_\varrho$ is a homeomorphism of the compact space $\hat{S}$ onto the compact spectrum of $l^1(S)$.

From now on we identify $\hat{S}$ with the spectrum of $l^1(S)$ and the Gelfand transform of $f \in l^1(S)$ is the function $\hat{f} \in C(\hat{S}, \mathbb{R})$ defined by

$$\hat{f}(\varrho)=\sum_{s \in S} f(s)\varrho(s) \quad \text{for} \quad \varrho \in \hat{S}.$$

By the Stone-Weierstrass theorem we get the following

1.5. Corollary. *The set of Gelfand transforms $\{\hat{f} \mid f \in l^1(S)\}$ is a dense subalgebra of $C(\hat{S}, \mathbb{R})$.*

For a real Radon measure μ on $\hat{S}$ we introduce the transform $\hat{\mu}$ of μ as the function $\hat{\mu}: S \to \mathbb{R}$ given by

$$\hat{\mu}(s)=\int_{\hat{S}} \varrho(s)d\mu(\varrho) \quad \text{for} \quad s \in S.$$

1.6. Corollary. *The mapping $\mu \to \hat{\mu}$ is one-to-one.*

Proof. This follows from Corollary 1.5. together with the formula

$$\sum_{s \in S} f(s)\hat{\mu}(s)=\int_{\hat{S}} \hat{f}(\varrho)d\mu(\varrho). \qquad \square$$

§ 2. Positive Definite Functions

Inspired by the paper [5] of Ressel we make the following definition.

2.1. Definition. Let S be an abelian semigroup. A real-valued function $f: S \to \mathbb{R}$ is called *positive definite* if f is bounded and has the following property:

For every $n \in \mathbb{N}$ and for every n-tuple $(s_1, \ldots, s_n)$ of elements from S the $n \times n$-matrix

$$(f(s_i+s_j))_{i,j=1,\ldots,n}$$

is positive semi-definite.

The set of positive definite functions on S is a convex cone $\mathscr{P}=\mathscr{P}(S)$ in the vector space $\mathscr{F}(S)$ of all real-valued functions on S. The cone $\mathscr{P}$ is closed in the topology of pointwise convergence.

2.2. Proposition. *Every $f\in\mathscr{P}$ has the following properties:*

(i) $f(2s)\geqq 0$ *for* $s\in S$, *in particular* $f(0)\geqq 0$

(ii) $(f(s+t))^2\leqq f(2s)f(2t)$ *for* $s,t\in S$

(iii) $|f(s)|\leqq f(0)$ *for* $s\in S$.

Proof. (i) and (ii) follow immediately from the definition. As a special case of (ii) we get $(f(s))^2\leqq f(2s)f(0)$ for $s\in S$, so $f(0)=0$ implies that f is identically zero.

In the proof of (iii) we can therefore assume that $f(0)=1$ and hence we have

$$(f(s))^2\leqq f(2s) \quad \text{for all} \quad s\in S.$$

By iterated application of this inequality we get

$$(f(s))^{2^n}\leqq f(2^n s) \quad \text{for} \quad n\in\mathbb{N} \quad \text{and} \quad s\in S,$$

and since f is assumed to be bounded we conclude that $|f(s)|\leqq 1$ for all $s\in S$. □

We now define

$$\mathscr{P}_1=\{f\in\mathscr{P}\,|\,f(0)=1\}.$$

By Proposition 2.2 (iii) it is clear that $\mathscr{P}_1$ is a compact convex subset of $\mathscr{F}$ in the topology of pointwise convergence, and $\mathscr{P}_1$ is a base for the cone $\mathscr{P}$. The set of characters $\hat{S}$ is easily seen to be a closed subset of $\mathscr{P}_1$.

The main result is the following:

2.3. Theorem. *The set $\mathscr{P}_1$ is a Choquet simplex and the set of extreme points of $\mathscr{P}_1$ is $\hat{S}$.*

In the proof we need the following Lemma.

2.4. Lemma. *Let $f\in\mathscr{P}$ and $a\in S$ be given. Then the functions $f_1, f_2\in\mathscr{F}$ defined by*

$$f_1(s)=f(s)+f(s+a) \quad \text{and} \quad f_2(s)=f(s)-f(s+a)$$

both belong to $\mathscr{P}$.

Proof. Let $s_1,\dots,s_n\in S$ and $c_1,\dots,c_n\in\mathbb{R}$ be given and consider the function $F:S\to\mathbb{R}$ defined by

$$F(s)=\sum_{i,j=1}^{n} f(s+s_i+s_j)c_ic_j.$$

We claim that $F\in\mathscr{P}$. In fact, it is certainly bounded, and for $t_1,\dots,t_m\in S$ and $d_1,\dots,d_m\in\mathbb{R}$ we have

$$\sum_{p,q=1}^{m} F(t_p+t_q)d_pd_q=\sum_{p,q=1}^{m}\sum_{i,j=1}^{n} f(t_p+t_q+s_i+s_j)c_ic_jd_pd_q$$

which is non-negative by the definition applied to the finite set of elements of $S: t_p+s_i$, $p=1,\dots,m$; $i=1,\dots,n$ and the corresponding set of numbers: d_pc_i.

In particular $|F(a)| \leqq F(0)$ for $a \in S$, i.e.

$$\sum_{i,j=1}^{n} (f(s_i+s_j) \pm f(a+s_i+s_j))c_i c_j \geqq 0 \quad \text{for} \quad a \in S. \qquad \square$$

Proof of Theorem 2.3. Let $f \in \mathscr{P}_1$ be an extreme element of $\mathscr{P}_1$ and let $a \in S$ be arbitrary. By Lemma 2.4 it follows immediately that there exists a constant $\lambda = \lambda(a) \geqq 0$ such that

$$f(s) + f(s+a) = \lambda f(s) \quad \text{for} \quad s \in S.$$

and hence

$$\lambda = 1 + f(a)$$

so that

$$f(s+a) = f(a) f(s) \quad \text{for} \quad s \in S$$

which shows that $f \in \hat{S}$.

Conversely we shall now prove that every $\varrho \in \hat{S}$ is an extreme point of $\mathscr{P}_1$. To $\varrho \in \hat{S}$ and $a \in S$ we define a function $e_a : \mathscr{P}_1 \to \mathbb{R}$ by

$$e_a(f) = f(2a) - 2f(a)\varrho(a) + (\varrho(a))^2 \quad \text{for} \quad f \in \mathscr{P}_1.$$

Then e_a is continuous and affine, $e_a(\varrho) = 0$ and

$$e_a(\varphi) = (\varphi(a) - \varrho(a))^2 \geqq 0 \quad \text{for every} \quad \varphi \in \hat{S},$$

and since $\mathrm{ext}(\mathscr{P}_1) \subseteqq \hat{S}$, it follows that $e_a \geqq 0$ on $\mathrm{ext}(\mathscr{P}_1)$, hence on $\mathscr{P}_1$.

We next define a function $E : \mathscr{P}_1 \to [0, \infty]$ by

$$E(f) = \sup\left\{ \sum_{i=1}^{n} e_{a_i}(f) \,\middle|\, n \in \mathbb{N}, a_1, \ldots, a_n \in S \right\}.$$

As the supremum of an upper filtering family of continuous affine functions, E is lower semicontinuous and concave and it attains therefore its infimum over $\mathscr{P}_1$ in an extreme point ϱ_0 of $\mathscr{P}_1$. Since $E \geqq 0$ and $E(\varrho) = 0$ we have $E(\varrho_0) = 0$, hence $e_a(\varrho_0) = 0$ for all $a \in S$. But $\varrho_0 \in \hat{S}$, so finally

$$(\varrho_0(a) - \varrho(a))^2 = 0 \quad \text{for all} \quad a \in S,$$

which proves that $\varrho = \varrho_0$ is extreme.

By the theorem of Krein-Milman follows now that every $f \in \mathscr{P}_1$ is the barycenter of a probability measure μ on the compact set $\hat{S}$, hence

$$f(s) = \int_{\hat{S}} \varrho(s) d\mu(\varrho) \quad \text{for} \quad s \in S.$$

A Radon measure μ on $\hat{S}$ verifying this equality is by Corollary 1.6 uniquely determined. The set $\mathscr{P}_1$ thus being affinely isomorphic to the simplex of probability measures on the compact set $\hat{S}$, we have proved that $\mathscr{P}_1$ is a simplex. $\square$

2.5. Corollary. *For every $f \in \mathscr{P}$ there exists one and only one positive Radon measure μ on $\hat{S}$ such that*

$$f(s) = \int_{\hat{S}} \varrho(s) d\mu(\varrho) \quad \textit{for} \quad s \in S.$$

2.6. Remarks. 1) The vector space of real Radon measures on $\hat{S}$ is denoted $M(\hat{S})$. By Corollary 1.6. the transformation $\hat{}$ is a one-to-one mapping of $M(\hat{S})$ into $\mathscr{F}(S)$ and $\hat{}$ is a bijection of $M_+(\hat{S})$ onto $\mathscr{P}(S)$. It is furthermore easy to verify that $\hat{}$ is a homeomorphism from $M_+(\hat{S})$ with the vague topology onto $\mathscr{P}(S)$ with the topology of pointwise convergence.

The semigroup structure on $\hat{S}$ induces a convolution $*$ on $M(\hat{S})$

$$\langle\mu * \nu, g\rangle = \int_{\hat{S}^2} g(\varrho\varphi)d\mu\otimes\nu(\varrho,\varphi) \quad \text{for} \quad g\in C(\hat{S},\mathbb{R}),$$

and we clearly have $(\mu*\nu)\hat{} = \hat{\mu}\hat{\nu}$. The cone $\mathscr{P}(S)$ is stable under pointwise multiplication.

2) Let $\varphi: S\to\mathbb{R}$ be a bounded function and $L_\varphi: l^1(S)\to\mathbb{R}$ the associated linear form

$$L_\varphi(f) = \sum_{s\in S} f(s)\varphi(s).$$

Then it is easy to see that φ is positive definite if and only if $L_\varphi(f*f)\geqq 0$ for all $f\in l^1(S)$.

3) The proof given above that $\varrho\in\hat{S}$ is extreme in $\mathscr{P}_1$ is based on lectures by Heinz Bauer, cf. [0]. Another proof follows easily from the general integral representation for compact convex sets together with the uniqueness in Corollary 1.6.

4) Suppose that S is an abelian group. Then every $\varrho\in\hat{S}$ is a group character (assuming only the values 1 and -1), and therefore every $f\in\mathscr{P}$ is positive definite in the group sense. Furthermore, putting $t=-s$ in Proposition 2.2 (ii), it follows that $f\in\mathscr{P}$ satisfies $f(2s)=f(0)$ for every $s\in S$. Therefore, if S is 2-divisible (i.e. every $s\in S$ is of the form $2t$ for some $t\in S$) every $f\in\mathscr{P}$ is a non-negative constant.

The characteristic function $1_{\{0\}}$ is positive definite in the group sense (cf. [1]), but $1_{\{0\}}\in\mathscr{P}$ if and only if $2s=0$ for all $s\in S$. It follows that the notion of positive definiteness in the sense of Definition 2.1 coincides with the notion of a real-valued positive definite function in the group sense precisely for abelian groups where every element has order two.

§ 3. Negative Definite Functions

In analogy with the group case (cf. Berg, Forst [1]) we introduce a class of functions on S called negative definite.

3.1. Definition. Let S be an abelian semigroup. A function $\psi: S\to[0,\infty[$ is called *negative definite* if for every $n\in\mathbb{N}$ and for every n-tuple $(s_1,\dots,s_n)$ of elements from S the $n\times n$-matrix

$$(\psi(s_i)+\psi(s_j)-\psi(s_i+s_j))_{i,j=1,\dots,n}$$

is positive semi-definite.

The set of all negative definite functions is a closed convex cone $\mathscr{N}$ in $\mathscr{F}$.

3.2. Proposition. *For a function* $\psi: S \to [0, \infty[$ *the following conditions are equivalent:*

(i) $\psi \in \mathcal{N}$.

(ii) $e^{-t\psi} \in \mathcal{P}$ *for every* $t>0$.

(iii) *For every* $n \geqq 2$, *every n-tuple* $(s_1, \ldots, s_n)$ *of elements from S and every n-tuple* $(c_1, \ldots, c_n)$ *of real numbers such that* $\sum_{i=1}^{n} c_i = 0$ *we have*

$$\sum_{i,j=1}^{n} \psi(s_i + s_j) c_i c_j \leqq 0 .$$

Proof. (i)⇒(ii). It is enough to prove that $e^{-\psi} \in \mathcal{P}$ if $\psi \in \mathcal{N}$. Let $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{R}$ be given. Since the matrix $(\psi(s_i) + \psi(s_j) - \psi(s_i + s_j))$ is positive semi-definite, so is the matrix $(\exp(\psi(s_i) + \psi(s_j) - \psi(s_i + s_j)))$, and therefore

$$\sum_{i,j=1}^{n} \exp(-\psi(s_i + s_j)) c_i c_j = \sum_{i,j=1}^{n} \exp(\psi(s_i) + \psi(s_j) - \psi(s_i + s_j)) d_i d_j \geqq 0$$

where $d_i = \exp(-\psi(s_i)) c_i$.

(ii)⇒(iii). With s_i and c_i as in (iii) we have

$$\sum_{i,j=1}^{n} \frac{1}{t}(1 - \exp(-t\psi(s_i + s_j))) c_i c_j = -\frac{1}{t} \sum_{i,j=1}^{n} \exp(-t\psi(s_i + s_j)) \leqq 0$$

and letting $t \to 0$ we get

$$\sum_{i,j=1}^{n} \psi(s_i + s_j) c_i c_j \leqq 0 .$$

(iii)⇒(i). Let $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{R}$ be given, and consider the $(n+1)$-tuples $(0, s_1, \ldots, s_n)$ and $(c, c_1, \ldots, c_n)$, where $c = -\sum_{i=1}^{n} c_i$. By (iii) we then have

$$c^2 \psi(0) + 2 \sum_{i=1}^{n} \psi(s_i) c_i c + \sum_{i,j=1}^{n} \psi(s_i + s_j) c_i c_j \leqq 0 ,$$

and consequently

$$\sum_{i,j=1}^{n} (\psi(s_i) + \psi(s_j) - \psi(s_i + s_j)) c_i c_j \geqq c^2 \psi(0) \geqq 0 .$$ □

3.3. Proposition.

a) *For* $\psi \in \mathcal{N}$ *we have* $\psi(s) \geqq \psi(0)$ *for all* $s \in S$.

b) *For* $\psi \in \mathcal{N}$ *we have* $\psi - \psi(0) \in \mathcal{N}$.

c) *For* $\varphi \in \mathcal{P}$ *we have* $\varphi(0) - \varphi \in \mathcal{N}$.

Proof. a) Let $\psi \in \mathcal{N}$ and consider the elements $0, s \in S$ and the pair of numbers $(1, -1)$. By Proposition 3.2 (iii) we then get

$$\psi(s) \geqq \tfrac{1}{2}(\psi(0) + \psi(2s)) .$$

Applying this inequality to $2s \in S$ we get

$$\psi(s) \geqq \left(\frac{1}{2}+\frac{1}{2^2}\right)\psi(0)+\frac{1}{2^2}\psi(2^2 s),$$

and by iteration

$$\psi(s) \geqq \left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)\psi(0)+\frac{1}{2^n}\psi(2^n s)$$

$$\geqq \left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)\psi(0).$$

For $n \to \infty$ we get $\psi(s) \geqq \psi(0)$.

The properties b) and c) follow immediately from Proposition 3.2. □

3.4. Corollary. *Every negative definite function ψ is the pointwise limit of negative definite functions of the form $c-\varphi$, where $\varphi \in \mathscr{P}$ and $c \geqq \varphi(0)$.*

Proof.

$$\psi(s)=\lim_{t \to 0}\frac{1}{t}(1-\exp(-t\psi(s)))\,. \qquad \square$$

The following property of negative definite functions is crucial in the proof of the "Lévy-Khinchin"-representation we will obtain in Theorem 3.7 below.

3.5. Proposition. *Every $\psi \in \mathscr{N}$ satisfies the inequalities*

$$|\psi(s)-\psi(t)| \leqq \psi(s+t) \leqq \psi(s)+\psi(t) \quad \textit{for} \quad s,t \in S\,.$$

Proof. The set of functions $\psi \in \mathscr{N}$ verifying the above inequalities is a closed convex cone C containing the non-negative constants. In order to prove $C=\mathscr{N}$ is suffices by Corollary 3.4 to prove that $1-f \in C$ for $f \in \mathscr{P}_1$, and by Theorem 2.3 it even suffices to show that $1-\varrho \in C$ for $\varrho \in \hat{S}$, but in this case the inequalities are straightforward. □

3.6. Proposition. *For $\psi \in \mathscr{N}$ and $a \in S$ the function $\Delta_a\psi(s)=\psi(s+a)-\psi(s)$ is positive definite.*

Proof. By Proposition 3.5 follows that $|\Delta_a\psi(s)| \leqq \psi(a)$ for all $s \in S$, so $\Delta_a\psi$ is bounded.

By Corollary 3.4 it suffices to prove the positive definiteness of $\Delta_a\psi$ for $\psi \in \mathscr{N}$ of the form $\psi=c-\varphi$, where $c \geqq \varphi(0)$ and $\varphi \in \mathscr{P}$, but this amounts to proving $s \to \varphi(s)-\varphi(a+s)$ is positive definite, which is known from Lemma 2.4. □

We now enumerate the three types of negative definite functions which will occur in the "Lévy-Khinchin" formula (Theorem 3.7 below):

1) The non-negative constant functions.

2) Functions $h: S \to [0,\infty[$ satisfying $h(s+t)=h(s)+h(t)$ for all $s,t \in S$.

3) Functions of the form $\psi(s)=\int_{\hat{S}\setminus\{1\}}(1-\varrho(s))d\mu(\varrho)$ where μ is a non-negative Radon measure on $\hat{S}\setminus\{1\}$ for which the integral on the right-hand side is finite for every $s \in S$.

It is easy to verify that functions of these types, and consequently sums of such functions, are negative definite.

3.7. Theorem. *Let $\psi\in\mathcal{N}$. Then there exist*

1) *a non-negative constant c,*
2) *a function $h: S\to[0,\infty[$ satisfying $h(s+t)=h(s)+h(t)$ for $s,t\in S$,*
3) *a non-negative Radon measure μ on $\hat{S}\setminus\{1\}$ with the property that*

$$\int_{\hat{S}\setminus\{1\}}(1-\varrho(s))d\mu(\varrho)<\infty \quad \textit{for all} \quad s\in S,$$

such that

$$(*)\quad \psi(s)=c+h(s)+\int_{\hat{S}\setminus\{1\}}(1-\varrho(s))d\mu(\varrho) \quad \textit{for} \quad s\in S.$$

The triple (c,h,μ) is uniquely determined by ψ and given in the following way:

$$c=\psi(0)$$

$$h(s)=\lim_{n\to\infty}\frac{\psi(ns)}{n} \quad \textit{for} \quad s\in S$$

$$d\mu(\varrho)=(1-\varrho(a))^{-1}d\sigma_a(\varrho) \quad \textit{on the open set} \quad \{\varrho\in\hat{S}\,|\,\varrho(a)<1\},$$

where σ_a is the finite positive measure on $\hat{S}$ satisfying $\hat{\sigma}_a=\Delta_a\psi$.

The measure μ in the above representation is called the *Lévy measure* for ψ.

Proof. The uniqueness of the representation: Suppose ψ has a representation of the form $(*)$. Then clearly $c=\psi(0)$ and

$$\frac{\psi(ns)}{n}=\frac{c}{n}+h(s)+\int_{\hat{S}\setminus\{1\}}\frac{1}{n}(1-(\varrho(s))^n)d\mu(\varrho) \quad \text{for} \quad s\in S \quad \text{and} \quad n\in\mathbb{N}.$$

Since

$$0\leqq\frac{1}{n}(1-(\varrho(s))^n)\leqq 1-\varrho(s),$$

the dominated convergence theorem can be applied to the effect that

$$h(s)=\lim_{n\to\infty}\frac{\psi(ns)}{n}.$$

For $a\in S$ we find

$$\Delta_a\psi=h(a)+\int_{\hat{S}\setminus\{1\}}\varrho(s)(1-\varrho(a))d\mu(\varrho)=\hat{\sigma}_a(s)$$

which shows that $\sigma_a=h(a)\varepsilon_1+(1-\varrho(a))\mu$, hence $\mu=(1-\varrho(a))^{-1}\sigma_a$ on the open set $\mathcal{O}_a=\{\varrho\in\hat{S}\,|\,\varrho(a)<1\}$, and the open sets $(\mathcal{O}_a)_{a\in S}$ evidently form a covering of $\hat{S}\setminus\{1\}$. This proves that μ is uniquely determined.

The existence of the representation: Let $\psi\in\mathcal{N}$. We will define the measure μ by the formula

$$\mu=(1-\varrho(a))^{-1}\sigma_a \quad \text{on} \quad \mathcal{O}_a,$$

so we need the following compatibility assertion:

$$(**) \quad (1-\varrho(a))^{-1}\sigma_a = (1-\varrho(b))^{-1}\sigma_b \quad \text{in} \quad \mathcal{O}_a \cap \mathcal{O}_b \quad \text{for} \quad a, b \in S.$$

To see this we remark that

$$\begin{aligned} &\psi(s+a)+\psi(s+b)-\psi(s)-\psi(s+a+b) \\ &= \int_{\hat{S}} \varrho(s)(1-\varrho(b))d\sigma_a(\varrho) = \int_{\hat{S}} \varrho(s)(1-\varrho(a))d\sigma_b(\varrho) \quad \text{for} \quad s \in S, \end{aligned}$$

hence

$$(1-\varrho(b))d\sigma_a(\varrho) = (1-\varrho(a))d\sigma_b(\varrho)$$

by Corollary 1.6, and (**) follows.

It is easy to see that the measure μ defined above satisfies

$$(1-\varrho(a))d\mu(\varrho) = d\sigma_a(\varrho)|\hat{S}\setminus\{1\} \quad \text{for all} \quad a \in S$$

and since σ_a is a finite measure for every $a \in S$, μ has the required integrability property. We can therefore define the function

$$h(s) = \psi(s) - \psi(0) - \int_{\hat{S}\setminus\{1\}} (1-\varrho(s))d\mu(\varrho) \quad \text{for} \quad s \in S$$

which is non-negative, since

$$\int_{\hat{S}\setminus\{1\}} (1-\varrho(s))d\mu(\varrho) \leqq \sigma_s(\hat{S}) = \psi(s) - \psi(0).$$

For $a \in S$ we find

$$\begin{aligned} \Delta_a h(s) &= \Delta_a \psi(s) - \int \varrho(s)(1-\varrho(a))d\mu(\varrho) \\ &= [\sigma_a - (1-\varrho(a))\mu]^\wedge(s) = \sigma_a(\{1\}), \end{aligned}$$

because $\sigma_a - (1-\varrho(a))\mu$ is concentrated in the point 1. The function $\Delta_a h$ being constant we have in particular

$$\Delta_a h(s) = \Delta_a h(0) \quad \text{for} \quad s \in S,$$

hence

$$h(a+s) = h(s) + h(a) - h(0)$$

but $h(0) = 0$. □

From the proof of existence in Theorem 3.7 we can extract the following result, which will be useful later.

3.8. Proposition. *Let $\psi: S \to [0, \infty[$ be a function with the property that $\Delta_a \psi$ is positive definite for every $a \in S$. Then ψ has a representation of the form (*) and in particular ψ is negative definite.*

3.9. Remark. In analogy with the theory of convolution semigroups on groups, cf. Berg, Forst [1], the following results are not difficult to obtain:

By a *convolution semigroup* on $\hat{S}$ we mean a family $(\mu_t)_{t>0}$ of positive measures on $\hat{S}$ satisfying

1) $\mu_t(\hat{S}) \leqq 1$ for $t>0$.
2) $\mu_t * \mu_s = \mu_{t+s}$ for $t, s>0$
3) $\mu_t \to \varepsilon_1$ vaguely for $t \to 0$.

There is a bijection $(\mu_t)_{t>0} \leftrightarrow \psi$ between the set of convolution semigroups on $\hat{S}$ and the set $\mathcal{N}$ established by

$$\hat{\mu}_t(a) = e^{-t\psi(a)} \quad \text{for} \quad t>0, \quad a \in S.$$

The Lévy measure μ for $\psi \leftrightarrow (\mu_t)_{t>0}$ is equal to

$$\mu = \lim_{t \to 0} \frac{1}{t} (\mu_t | \hat{S} \setminus \{1\}),$$

where the limit is in the vague topology for the measures on the locally compact space $\hat{S} \setminus \{1\}$.

3.10. Proposition. *Suppose $\psi \in \mathcal{N}$ is a bounded function. Then the representation* (*) *takes the form*

$$\psi(s) = \psi(0) + \int_{\hat{S} \setminus \{1\}} (1 - \varrho(s)) d\mu(\varrho),$$

and the Lévy measure μ is finite. In particular ψ has the form $c - \varphi$ where $\varphi \in \mathscr{P}$ and $c \geqq \varphi(0)$.

Proof. Since ψ is bounded we clearly have

$$h(s) = \lim_{n \to \infty} \frac{\psi(ns)}{n} = 0 \quad \text{for all} \quad s \in S$$

and there exists a constant α such that

$$\int (1 - \varrho(s)) d\mu(\varrho) \leqq \alpha \quad \text{for all} \quad s \in S.$$

For $f \in l^1_+(S)$ with $\sum_{s \in S} f(s) = 1$ we get

$$(***) \quad \sum_{s \in S} f(s) \int (1 - \varrho(s)) d\mu(\varrho) = \int (1 - \hat{f}(\varrho)) d\mu(\varrho) \leqq \alpha.$$

Let K be an arbitrary compact subset of $\hat{S} \setminus \{1\}$. For $\varrho \in K$ there exists $s \in S$ such that $\varrho(s) < 1$ and therefore $\hat{\varepsilon}_s < 1$ in an open neighbourhood U_ϱ of ϱ in $\hat{S} \setminus \{1\}$. By a compactness argument there exist finitely many points $s_1, \ldots, s_n \in S$ such that

$$f = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{s_i}$$

satisfies $|\hat{f}| \leqq 1$ and $\hat{f} < 1$ in a neighbourhood of K. All the functions $g = f^{*(2p+1)}$, $p \in \mathbb{N}$, satisfy $g \in l^1_+(S)$ and $\sum_{s \in S} g(s) = 1$, and (***) therefore gives

$$\int_{\hat{S} \setminus \{1\}} (1 - (\hat{f}(\varrho))^{2p+1}) d\mu(\varrho) \leqq \alpha \quad \text{for all} \quad p \in \mathbb{N}.$$

By the Fatou lemma we then get

$$\mu(K) \leqq \int_{\hat{S}\setminus\{1\}} \liminf_{p\to\infty} (1-(\hat{f}(\varrho))^{2^{p+1}})d\mu(\varrho) \leqq \alpha,$$

and hence $\mu(\hat{S}\setminus\{1\}) \leqq \alpha$, which proves that μ is finite. □

§ 4. Relation to Monotone and Alternating Functions of Infinite Order

For a function $f:S\to\mathbb{R}$ and an element $a_1\in S$ we define a function $\nabla_1 f(\cdot\,; a_1):S\to\mathbb{R}$ by

$$\nabla_1 f(s; a_1) = f(s) - f(s+a_1) \quad \text{for} \quad s\in S.$$

For given elements $a_1, \ldots, a_n \in S$ we define inductively functions

$$\nabla_p f(\cdot\,; a_1, \ldots, a_p): S\to\mathbb{R}, \quad \text{where} \quad p=2, \ldots, n:$$

$$\nabla_p f(s; a_1, \ldots, a_p) = \nabla_{p-1} f(s; a_1, \ldots, a_{p-1}) - \nabla_{p-1} f(s+a_p; a_1, \ldots, a_{p-1}), \qquad s\in S.$$

A function $f:S\to[0,\infty[$ is called *monotone* (resp. *alternating*) of *infinite order* (cf. Choquet [2]) if and only if

$$\nabla_n f(s; a_1, \ldots, a_n) \geqq 0 \; [\text{resp. } \nabla_n f(s; a_1, \ldots, a_n) \leqq 0]$$

for all $n\in\mathbb{N}$ and all $a_1, \ldots, a_n\in S$. The set of all monotone (resp. alternating) functions of infinite order is denoted by $\mathscr{M}$ (resp. $\mathscr{A}$). It is clear that $\mathscr{M}$ and $\mathscr{A}$ are convex cones.

4.1. Lemma. *A function $\psi:S\to[0,\infty[$ belongs to $\mathscr{A}$ if and only if for every $a\in S$ the function $\Delta_a\psi(s) = \psi(s+a) - \psi(s)$ belongs to $\mathscr{M}$.*

Proof. For $a, s, a_1, \ldots, a_n \in S$ we have the identity

$$\nabla_n(\Delta_a\psi)(s; a_1, \ldots, a_n) = -\nabla_{n+1}\psi(s; a_1, \ldots, a_n, a)$$

from which both directions can be derived easily. □

We recall the definition that a subcone E of a convex cone C is extreme iff $f\in E$, $f=f_1+f_2$, $f_1\in C$, $f_2\in C$ implies $f_1, f_2\in E$.

4.2. Theorem. a) *$\mathscr{M}$ is an extreme subcone of $\mathscr{P}$.* b) *$\mathscr{A}$ is an extreme subcone of $\mathscr{N}$.* c) *If S is 2-divisible (i.e. every element $s\in S$ is of the form $2t$ for some $t\in S$) then $\mathscr{M}=\mathscr{P}$ and $\mathscr{A}=\mathscr{N}$.*

Proof. a) It is easy to see that $\mathscr{M}_1 = \{f\in\mathscr{M} \mid f(0)=1\}$ is a compact convex base for the convex cone $\mathscr{M}$, considered as a subset of $\mathscr{F}$ in the topology of pointwise convergence. From Choquet [2] §46 it is known that the extreme points of $\mathscr{M}_1$ are precisely the non-negative characters on S, hence $\mathscr{M}\subseteqq\mathscr{P}$ by Theorem 2.3. The unicity of the integral representation implies that $\mathscr{M}$ is extreme in $\mathscr{P}$.

b) The above Lemma 4.1 together with Proposition 3.8 show that $\mathscr{A}\subseteqq\mathscr{N}$. Suppose $f=f_1+f_2$ where $f_1, f_2\in\mathscr{N}$ and $f\in\mathscr{A}$. Then for any $a\in S$ we have

$$\Delta_a f = \Delta_a f_1 + \Delta_a f_2 \in \mathscr{M},$$

hence $\Delta_a f_1$, $\Delta_a f_2\in\mathscr{M}$, and again the above lemma gives $f_1, f_2\in\mathscr{A}$.

c) Every character is automatically non-negative if S is 2-divisible, so $\mathcal{M}_1$ and $\mathcal{P}_1$ have the same extreme points and consequently $\mathcal{M}=\mathcal{P}$.

It is easy to verify that non-negative constants, non-negative additive functions and functions of the type $1-\varrho$, where ϱ is a non-negative character on S, all belong to $\mathcal{A}$. By Theorem 3.7 we may now conclude that $\mathcal{N}\subseteqq\mathcal{A}$ in the 2-divisible case, hence $\mathcal{N}=\mathcal{A}$. □

4.3. Remarks. 1) If S is not 2-divisible, it may happen that $\mathcal{M}\neq\mathcal{P}$. An example is given by $S=\mathbb{N}_0=\{0,1,2,\ldots\}$ with addition. Here $n\to(-1)^n$ belongs to $\hat{S}\setminus\mathcal{M}$. Even $\mathcal{P}_+$ is different from $\mathcal{M}$ as the following function shows: $n\to\frac{1}{2}+\frac{1}{2}(-1)^n$.

2) It might happen that $\mathcal{M}=\mathcal{P}$ without S being 2-divisible. Let $S=\{0,a,b\}$ be the commutative semigroup with neutral element 0 and where $a+a=a+b=b+b=a$. Then S is not 2-divisible, but every character is easily seen to be non-negative so that $\mathcal{M}=\mathcal{P}$.

3) If S is an abelian group, every character assumes only the values 1 and -1 and therefore $\mathcal{M}$ is the non-negative constants. If S is not 2-divisible then $\mathcal{M}\neq\mathcal{P}$.

In fact, if S is not 2-divisible the abelian group $G=S/S_2$, where $S_2=\{2s|s\in S\}$, contains at least two elements, and all elements $g\in G$ satisfy $2g=0$. These two properties imply that there exists a homomorphism $\varphi:G\to\{-1,1\}$ onto the multiplicative group $\{-1,1\}$. (Apply Zorn's lemma, if G is infinite.)

An alternating function of infinite order is negative definite by Theorem 4.2 and hence has the unique representation given in Theorem 3.7. It is natural to guess that the measure should be concentrated on $(\hat{S}\setminus\{1\})_+$ in this case. This is in fact true:

4.4. Theorem. *Let $\psi\in\mathcal{N}$ have the representation*

$$\psi(s)=c+h(s)+\int_{\hat{S}\setminus\{1\}}(1-\varrho(s))d\mu(\varrho)$$

given by Theorem 3.7. Then $\psi\in\mathcal{A}$ if and only if the measure μ is concentrated on $(\hat{S}\setminus\{1\})_+$.

Proof. Let $T=(\hat{S}\setminus\{1\})\setminus(\hat{S}\setminus\{1\})_+$. We know already that $\mu(T)=0$ implies $\psi\in\mathcal{A}$ (cf. the proof of Theorem 4.2). Let us now assume that $\psi\in\mathcal{A}$. Since $\mathcal{A}$ is an extreme subcone of $\mathcal{N}$ we conclude that $\varphi_a(s)=\int_{T_a}(1-\varrho(s))d\mu(\varrho)$ belongs to $\mathcal{A}$, too, where $T_a=\{\varrho\in\hat{S}|\varrho(a)<0\}$. By Lemma 4.1 $\Delta_a\varphi_a$ is in $\mathcal{M}$, hence

$$0\leqq\varphi_a(2a)-\varphi_a(a)=\int_{T_a}\varrho(a)(1-\varrho(a))d\mu(\varrho)\,.$$

But the integrand on the right hand side is negative which shows that $\mu(T_a)=0$. Using that T_a is open for every $a\in S$ and that $T=\bigcup_{a\in S}T_a$ we get $\mu(T)=0$. □

Looking to Theorem 4.2 and to Proposition 3.2 we can state, that on a 2-divisible semigroup a non-negative function ψ is alternating of infinite order if and only if $e^{-t\psi}$ is monotone of infinite order for each $t>0$. We shall now apply the representation of Theorem 4.4 to show that this result is true in general.

4.5. Theorem. *Let ψ be a non-negative function on S. Then $\psi \in \mathscr{A}$ if and only if $e^{-t\psi} \in \mathscr{M}$ for each $t > 0$.*

Proof. If $e^{-t\psi} \in \mathscr{M}$ for all $t > 0$ then $1 - e^{-t\psi} \in \mathscr{A}$ and so $\psi = \lim_{t \to 0} t^{-1}(1 - e^{-t\psi}) \in \mathscr{A}$. Now assume ψ to be in $\mathscr{A}$. Then ψ has the representation given in Theorem 4.4. It is enough to show that $e^{-\psi}$ lies in $\mathscr{M}$. Observing that $\mathscr{M}$ is closed under pointwise multiplication (this follows from the integral representation for functions in $\mathscr{M}$) we are left with the problem to prove that

$$s \to \exp\left[-\int_{(\hat{S} \setminus \{1\})_+} (1 - \varrho(s)) d\mu(\varrho)\right]$$

belongs to $\mathscr{M}$, $e^{-h(s)}$ being a non-negative character on S. Approximating μ by finite measures and approximating these by measures with finite support the problem reduces to the following question: does $\varrho \in \hat{S}_+$ and $\alpha \in [0, \infty[$ imply that $e^{\alpha\varrho} \in \mathscr{M}$? This in fact is true, because $\alpha\varrho \in \mathscr{M}$ and

$$e^{\alpha\varrho(s)} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha\varrho(s))^n . \qquad \square$$

The cone $\mathscr{M}$ is with respect to pointwise multiplication a subsemigroup of $\mathscr{P}_+$. A natural question is the following: if a finite product of elements of $\mathscr{P}_+$ belongs to $\mathscr{M}$, do then the factors already belong to $\mathscr{M}$, too? In this generality it is not true; take as a counterexample $S = \mathbb{N}_0^2$ (with addition) and the non-negative positive definite functions $f = 1_{(2\mathbb{N}_0) \times \{0\}}$, $g = 1_{\{0\} \times (2\mathbb{N}_0)}$ whose product is the non-negative character $1_{\{(0,0)\}}$. A partial answer to the problem is given by

4.6. Proposition. a) *Let f and g be strictly positive elements in $\mathscr{P}$, i.e. assume that $f(s) > 0$ and $g(s) > 0$ for all $s \in S$. Then $fg \in \mathscr{M}$ implies $f \in \mathscr{M}$ and $g \in \mathscr{M}$.*

b) *Let $f \in \mathscr{P}_+$ have the property that for some integer $k \geqq 1$ f^k belongs to $\mathscr{M}$, then $f \in \mathscr{M}$.*

Proof. a) We shall show that $f \in \mathscr{M}$. Let $f(s) = \int_{\hat{S}} \varrho(s) d\mu(\varrho)$ be the integral representation of f. Then $f(s)g(s) = \int \varrho(s)g(s) d\mu(\varrho)$ and we may claim, $\mathscr{M}$ being an extreme subcone of $\mathscr{P}$, that

$$s \to \int_{T_a} \varrho(s)g(s) d\mu(\varrho)$$

belongs to $\mathscr{M}$, too, where again $T_a = \{\varrho \in \hat{S} | \varrho(a) < 0\}$, $a \in S$. Therefore this function is non-negative, hence

$$0 \leqq g(a) \int_{T_a} \varrho(a) d\mu(\varrho)$$

which is only possible if $\mu(T_a) = 0$. This shows that μ is concentrated on $\hat{S}_+$ and that consequently $f \in \mathscr{M}$.

b) Let $f(s) = \int \varrho(s) d\mu(\varrho)$ be the integral representation of f. Suppose that $f \in \mathscr{P}_+$ and $f^k \in \mathscr{M}$ for some $k \geqq 1$. If $f(a) = 0$, then necessarily also $f(a+s) = 0$

for all $s \in S$, in particular $f(2a)=0$. But $f(2a)=\int (\varrho(a))^2 d\mu(\varrho)$, so we get that $\mu(\{\varrho \in \hat{S} | \varrho(a) \neq 0\})=0$ and hence $\mu(T_a)=0$. If $f(a)>0$ then we use in analogy with a) that

$$s \to \int_{T_a} \varrho(s)\,(f(s))^{k-1} d\mu(\varrho)$$

belongs to $\mathscr{M}$, and this gives us as before that $\mu(T_a)=0$. Hence $f \in \mathscr{M}$. □

§ 5. Infinitely Divisible Positive Definite Functions

5.1. Definition. A function $f \in \mathscr{P}_1$ is called *infinitely divisible* if and only if for each $n \in \mathbb{N}$ there exists $f_n \in \mathscr{P}_1$ such that $f=(f_n)^n$, i.e. $f(s)=(f_n(s))^n$ for all $s \in S$. We denote the set of all infinitely divisible f in $\mathscr{P}_1$ by $\mathscr{P}^i$.

Our aim is to identify the logarithms of the infinitely divisible functions in $\mathscr{P}_1$. The functions in $\mathscr{P}^i$ are non-negative, but they may assume the value 0. This is the reason that we introduce $\mathscr{N}_\infty$ to be the closure of $\{\varphi \in \mathscr{N} | \varphi(0)=0\}$ in $[0, \infty]^S$.

5.2. Theorem. *Let f be a non-negative function in $\mathscr{P}_1$ and put $\varphi=-\log f$. Then the following conditions are equivalent:*

(i) $f \in \mathscr{P}^i$.

(ii) $\varphi \in \mathscr{N}_\infty$.

(iii) $e^{-t\varphi} \in \mathscr{P}_1$ *for all* $t>0$.

Proof. (i)⇒(ii). The k'th root of f is unique, if k is an odd integer >0 so for those k we have $\exp\left(-\frac{1}{k}\varphi\right) \in \mathscr{P}_1$. This implies (cf. Proposition 3.3) $1-\exp\left(-\frac{1}{k}\varphi\right) \in \mathscr{N}$ so that $\varphi = \lim_{\substack{k\to\infty \\ k \text{ odd}}} k\left(1-\exp\left(-\frac{1}{k}\varphi\right)\right) \in \mathscr{N}_\infty$.

(ii)⇒(iii). Let (φ_α) be a net in $\mathscr{N}$ converging to φ. Then for any $t>0$ we get $e^{-t\varphi_\alpha} \to e^{-t\varphi}$ so that $e^{-t\varphi} \in \mathscr{P}_1$ by Proposition 3.2.

(iii)⇒(i) Take $t=\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. □

5.3. Remarks. 1) The above proof shows that $\mathscr{N}_\infty$ is in fact the monotone sequential closure of $\{\varphi \in \mathscr{N} | \varphi(0)=0\}$. It even shows that any $f \in \mathscr{P}^i$ is the decreasing limit of a sequence $f_n \in \mathscr{P}^i$ with $c_n=\inf\{f_n(s) | s \in S\}>0$ and $\{s \in S | f_n(s)=c_n\}=\{s \in S | f(s)=0\}$ for all $n \in \mathbb{N}$.

2) If $f \in \mathscr{P}^i$ then the uniquely determined non-negative roots are all positive definite; they also belong to $\mathscr{P}^i$.

3) Replacing $\mathscr{P}_1$ by $\mathscr{M}_1$, $\mathscr{P}^i$ by $\mathscr{M}^i$ and $\mathscr{N}_\infty$ by $\mathscr{A}_\infty$ ($\mathscr{M}^i$ and $\mathscr{A}_\infty$ defined in an obvious manner) the three conditions of Theorem 5.2 remain equivalent. In fact we have by Proposition 4.6 that $\mathscr{P}^i \cap \mathscr{M}_1 = \mathscr{M}^i$ which of course means that if an infinitely divisible positive definite function is monotone of infinite order, then all (positive) roots of this function are monotone of order infinity again.

Theorem 5.2 states that $\mathcal{N}_\infty$ is precisely the class of minus logarithms of infinitely divisible positive definite functions. We should like to have an integral representation for this class but for the time being we know it only for functions in $\mathcal{N}$ (Theorem 3.7) corresponding to the strictly positive elements in $\mathcal{P}^i$. A natural way to get out of this problem is to determine for a given $f\in\mathcal{P}^i$ the set $\{f>0\}$ where f is strictly positive. As it turns out that this subset always is a subsemigroup of S (containing the neutral element), we have in a certain sense solved our problem. But new questions arise: What are the subsemigroups of the special form $\{f>0\}$ for some $f\in\mathcal{P}^i$? The example $\mathbb{N}_0\subset[0,\infty[$ shows that there exist subsemigroups not belonging to this class. A further interesting question then is the following: Given a subsemigroup $T\subseteq S$ of the above mentioned type, characterise those strictly positive elements $f\in\mathcal{P}^i(T)$, for which their "zero-extension" $\tilde{f}$, defined by $\tilde{f}|T=f$ and $\tilde{f}|T^c=0$, belongs to $\mathcal{P}^i(S)$. We are far from giving completely satisfactory answers to these questions but nevertheless some positive results are available. We begin by stating some inequalities.

5.4. Lemma. *If $f\in\mathcal{P}^i$, then for all $s,t\in S$ we have*

a) $f(s+t)\leqq\sqrt{f(2s)f(2t)}$.

b) $f(s+2t)\leqq f(2t)$.

c) $f(s)f(t)\leqq f(s+t)$.

d) $f(s+t)f(t)\leqq f(s)$.

Proof. a) is a restatement of Proposition 2.2 (ii). b) follows from Proposition 2.2 (iii), $s\to f(s+2t)$ being positive definite. c) and d) are easily derived from Proposition 3.5, taking into account that $-\log f\in\mathcal{N}_\infty$ by Theorem 5.2. □

5.5. Proposition. *Let $\mathcal{T}=\{\{f>0\}|f\in\mathcal{P}^i\}$. Then we have*

a) *$T\in\mathcal{T}$ if and only if T is a subsemigroup of S containing the neutral element, such that 1_T is positive definite on S.*

b) *For every $T\in\mathcal{T}$ it is true that $T^c+T\subseteq T^c$ and $2s\in T^c$ implies $s+S\subseteq T^c$.*

c) *If S is 2-divisible, then even $T^c+S\subseteq T^c$ for all $T\in\mathcal{T}$.*

Proof. a) If $T=\{f>0\}$, $f\in\mathcal{P}^i$, then clearly $0\in T$. T is a subsemigroup by inequality c) of Lemma 5.4 and $1_T=\lim_{n\to\infty}\sqrt[n]{f}$ is positive definite. The other direction is trivial.

b) is an obvious consequence of the inequalities a) and d) in Lemma 5.4.

c) follows from inequality b) in the above Lemma. □

5.6. Corollary. *If S is 2-divisible, then for any $T\in\mathcal{T}$ and any $f\in\mathcal{P}(T)$ the zero-extension $\tilde{f}$ of f belongs to $\mathcal{P}(S)$.*

Proof. Let $s_1,\dots,s_n\in S$ and $c_1,\dots,c_n\in\mathbb{R}$ be given. Applying part c) of the previous proposition we get

$$\sum_{i,j=1}^{n}\tilde{f}(s_i+s_j)c_ic_j=\sum_{\{i|s_i\in T\}^2}f(s_i+s_j)c_ic_j\geqq 0\,. \qquad \square$$

It is not difficult to see, that the above corollary fails to be true on general semigroups. As an example let us consider the semigroup $\mathbb{N}_0=\{0,1,2,\dots\}$ under addition. It is easy to show that the characters are precisely the functions $n \to x^n$ for $x \in [-1, 1]$ and that $\hat{S}$ is homeomorphic to $[-1, 1]$.

5.7. Proposition. a) $\mathscr{T}(\mathbb{N}_0) = \{\mathbb{N}_0, 2\mathbb{N}_0, \{0\}\}$.

b) *The zero-extension of* $f: 2\mathbb{N}_0 \to \mathbb{R}$ *is positive definite if and only if* f *is monotone of infinite order on* $2\mathbb{N}_0$.

Proof. a) Assume that $T \in \mathscr{T}(\mathbb{N}_0)$ and that T is neither $\{0\}$ nor $\mathbb{N}_0$. Then of course $1 \notin T$ and $(2\mathbb{N}_0) \cap T^c = \emptyset$, because otherwise there would exist $k \in \mathbb{N}_0$ with $k + \mathbb{N}_0 \subseteqq T^c$ [by Proposition 5.5 (b)] which would imply that T is finite. Hence $2\mathbb{N}_0 \subseteqq T$ and from $T + T^c \subseteqq T^c$ we conclude $T = 2\mathbb{N}_0$.

b) Suppose that the zero-extension of f is positive definite. Call it $\tilde{f}$ and let $\tilde{f}(n) = \int_{-1}^{1} x^n d\mu(x)$ be its desintegration. Then

$$f(2n) = \tilde{f}(2n) = \int_{-1}^{1} x^{2n} d\mu(x) = \int_{0}^{1} y^n dv(y),$$

where v is the image measure of μ under $x \to x^2$. Hence $f \in \mathscr{M}(2\mathbb{N}_0)$.

Conversely assume that $f(2n) = \int_0^1 x^n dv(x)$. Let $n_1, \ldots, n_k \in \mathbb{N}_0$ and $c_1, \ldots, c_k \in \mathbb{R}$ be given. Put $N_1 = \{i | n_i \in 2\mathbb{N}_0\}$ and $N_2 = \{i | n_i \in 2\mathbb{N}_0 + 1\}$. Then

$$\sum_{i,j=1}^{k} \tilde{f}(n_i + n_j) c_i c_j = \sum_{N_1^2} f(n_i + n_j) c_i c_j + \sum_{N_2^2} f(n_i + n_j) c_i c_j.$$

The first sum on the right hand side is of course non-negative, and for the second we get

$$\sum_{N_2^2} f(n_i + n_j) c_i c_j = \sum_{N_2^2} \int_0^1 x^{(n_i + n_j)/2} dv(x) c_i c_j$$

$$= \int_0^1 \Big(\sum_{i \in N_2} c_i x^{n_i/2} \Big)^2 dv(x) \geqq 0.$$

Hence $\tilde{f} \in \mathscr{P}(\mathbb{N}_0)$. □

Without proof we mention, that $\mathscr{T}(\mathbb{N}_0^2)$ consists of the following ten semigroups:

$\mathbb{N}_0^2$, $(2\mathbb{N}_0) \times \mathbb{N}_0$, $\mathbb{N}_0 \times (2\mathbb{N}_0)$, $\mathbb{N}_0 \times \{0\}$, $\{0\} \times \mathbb{N}_0$, $(2\mathbb{N}_0) \times \{0\}$, $\{0\} \times (2\mathbb{N}_0)$, $\{(n_1, n_2) | n_1 + n_2 \in 2\mathbb{N}_0\}$, $(2\mathbb{N}_0) \times (2\mathbb{N}_0)$ and $\{0\}$.

Chapter II. Examples and Applications

§ 6. The Classical Laplace Transform

Let S be the additive semigroup $\mathbb{R}_+^p = \{x \in \mathbb{R}^p | x_i \geqq 0 \forall i = 1, \ldots, p\}$, where p is some positive integer. It is an elementary fact that the characters on this semigroup are given by exponentials; more precisely: the mapping

$$[0, \infty]^p \to \hat{\mathbb{R}}_+^p$$
$$t \to (s \to e^{-\langle s, t \rangle})$$

is a topological semigroup isomorphism. (We use the convention that $0\cdot\infty=0$ and $c\cdot\infty=\infty$ for $c>0$.) Corollary 2.5 now implies that

$$M_+([0,\infty]^p)\to\mathscr{P}(\mathbb{R}_+^p)$$
$$\mu\to\left(s\overset{\hat\mu}{\to}\int_{[0,\infty]^p} e^{-\langle s,t\rangle}d\mu(t)\right)$$

is a bijective map from the space of non-negative Radon measures on $[0,\infty]^p$ onto the cone of positive definite functions on $\mathbb{R}_+^p$.

This map is the classical Laplace Transformation. It is immediate that if μ is concentrated on $\mathbb{R}_+^p$, then its Laplace transform $\hat\mu$ is continuous. If on the other hand $\hat\mu$ is supposed to be continuous, then for any $j=1,\dots,p$ we get

$$\hat\mu(0,\dots,0,s_j,0,\dots,0)=1_{\{0\}}(s_j)\pi_j(\mu)(\{\infty\})+\int_{\mathbb{R}_+} e^{-s_jt}d\pi_j(\mu)(t),$$

where $\pi_j:[0,\infty]^p\to[0,\infty]$ is the projection on the j's coordinate, from which we may conclude that $\pi_j(\mu)(\{\infty\})=0$, hence $\mu([0,\infty]^p\setminus\mathbb{R}_+^p)=0$. Thus we get

6.1. Theorem. (cf. [5], Satz 1). *A real valued function on $\mathbb{R}_+^p$ is the Laplace transform of a finite non-negative measure on $\mathbb{R}_+^p$ if and only if it is continuous and positive definite.*

If $\psi:\mathbb{R}_+^p\to[0,\infty[$ is a negative definite function, then Theorem 3.7 gives us the unique representation

$$\psi(s)=\psi(0)+h(s)+\int_{[0,\infty]^p\setminus\{0\}}(1-e^{-\langle t,s\rangle})d\sigma(t)\quad\text{for all}\quad s\in\mathbb{R}_+^p$$

where h is additive and σ is a non-negative Radon measure on $[0,\infty]^p\setminus\{0\}$. Of course we can find a vector $t_0\in\mathbb{R}_+^p$ such that $h(s)=\langle t_0,s\rangle$ for all s, and then it is again not difficult to see that ψ is continuous if and only if σ is concentrated on $\mathbb{R}_+^p\setminus\{0\}$. Combining this with Theorem 5.2 we get, cf. [4] (Hirsch)

6.2. Theorem. *Let μ be a probability measure on $\mathbb{R}_+^p$, let f be its Laplace transform and put $\psi=-\log f$. Then the following properties are equivalent:*

(i) *μ is infinitely divisible (in the ordinary sense).*

(ii) *f is a continuous infinitely divisible positive definite function.*

(iii) *ψ is a continuous negative definite function.*

(iv) *$e^{-t\psi}$ is continuous and positive definite for all $t>0$.*

(v) *There exist $t_0\in\mathbb{R}_+^p$ and a non-negative Radon measure σ on $\mathbb{R}_+^p\setminus\{0\}$ such that*

$$\psi(s)=\langle t_0,s\rangle+\int_{\mathbb{R}_+^p\setminus\{0\}}(1-e^{-\langle t,s\rangle})d\sigma(t)\quad \textit{for all}\quad s\in\mathbb{R}_+^p.$$

6.3. Remark. A famous theorem of I.J. Schoenberg states the following (cf. [7]): A continuous function $f:\mathbb{R}_+\to\mathbb{C}$ has the property that $f\circ|\cdot|_n$ is positive semi-definite (in the group theoretic sense) on $\mathbb{R}^n$ for all $n \in \mathbb{N}$ ($|\cdot|$ denoting the euclidean norm) if and only if there exists a finite non-negative measure μ on $\mathbb{R}_+$ such that

$$f(t) = \int_0^\infty e^{-t^2\lambda} d\mu(\lambda) \quad \text{for all} \quad t \in \mathbb{R}_+ .$$

We only mention the fact that an application of Theorem 6.1 leads to a very easy and short proof of this result and refer the reader to [6].

§ 7. The Semigroup ([0, 1], $\wedge$)

Let us consider as a further example the unit interval $S=[0,1]$ with $s+t=s\wedge t=\min\{s,t\}$ as semigroup operation. Then S is a compact, metrisable, topological abelian semigroup with unit element 1. S consists only of idempotents, in particular it is 2-divisible. It is clear that characters on S can only assume the values 0 and 1, in fact we have

$$\hat{S} = \{1_{[a,1]} | 0 \leqq a \leqq 1\} \cup \{1_{]a,1]} | 0 \leqq a < 1\} .$$

7.1. Proposition. *A function $f: S \to \mathbb{R}$ is positive definite if and only if f is non-decreasing and non-negative.*

Proof. If f is positive definite, it has an integral representation $f(s) = \int_{\hat{S}} \varrho(s) d\mu(\varrho)$ and is hence non-decreasing, all characters having this property. For the converse direction we shall give a probabilistic proof. Assume that $f: S \to [0, \infty[$ is non-decreasing and let $0 \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_n \leqq 1$ and $c_1, \ldots, c_n \in \mathbb{R}$ be given. There exist a probability space and independent normally distributed random variables $X_1, X_2, \ldots, X_n$ with mean zero and variances $f(t_1), f(t_2)-f(t_1), \ldots, f(t_n)-f(t_{n-1})$. Put $Y_k = \sum_{i=1}^{k} X_i$; then if $i \leqq j$,

$$\begin{aligned} \mathbb{E}(Y_i Y_j) = \mathbb{E}(Y_i^2) &= f(t_1) + f(t_2) - f(t_1) + \ldots + f(t_i) - f(t_{i-1}) \\ &= f(t_i) = f(t_i \wedge t_j) \end{aligned}$$

implying

$$\sum_{i,j=1}^{n} f(t_i \wedge t_j) c_i c_j = \mathbb{E}\left(\sum_{i=1}^{n} c_i Y_i\right)^2 \geqq 0 . \qquad \square$$

7.2. Corollary. a) *$\psi: S \to [0, \infty[$ is negative definite if and only if ψ is non-increasing.*
b) *Every positive definite function on S is infinitely divisible.*

Proof. a) follows from Proposition 3.2 and b) is evident. □

7.3. Remark. If f is positive definite on S, its integral representation can be obtained in the following way: assume (without loss of generality) that $f(1)=1$. Put $S_a = \{s \in S | f(s) \geqq a\}$ for $0 \leqq a \leqq 1$. Then $1 \in S_a$ and $1_{S_a} \in \hat{S}$. The mapping $\Phi: [0, 1] \to \hat{S}$,

$\Phi(a)=1_{S_a}$ is easily seen to be (Borel to Baire) measurable. Let m be the Lebesgue measure on $[0,1]$ and μ its image under Φ. Then

$$f(s)=\int_0^1 1_{[0,f(s)]}(a)dm(a)=\int_0^1 1_{S_a}(s)dm(a)=\int_{\hat{S}}\varrho(s)d\mu(\varrho)$$

for all $s\in S$, which shows, that μ is the (unique) measure representing f.

§ 8. The Semigroup $L_1^\infty([0,1])$

In the example given above our semigroup was compact, metrisable and moreover a topological semigroup, which means that the semigroup operation is simultaneously continuous. And we can see that if a positive definite function is continuous in the neutral element, then almost every character in its desintegration has the same property. This of course holds also for the classical Laplace transform, cf. §6. We do not know whether this is true in general. The following exampel shows that if only separate continuity of the semigroup operation is assumed (i.e. the semigroup is "semitopological"), then the situation may be pathological.

Let S be the unit ball $L_1^\infty([0,1])$ in L^∞ over the unit interval with Lebesgue measure m (we only consider real valued functions). With multiplication and the $\sigma(L^\infty,L^1)$-topology S is a semitopological abelian semigroup with neutral element 1 and moreover compact and metrisable as a topological space. The function $\varphi:S\to\mathbb{R}$ defined by

$$\varphi(f)=\int_0^1 f(x)dm(x)$$

is a continuous positive definite function on S with $\varphi(1)=1$.

8.1. Theorem. *The unique probability measure giving the desintegration of φ is concentrated on a compact subset of $\hat{S}$ which contains no characters continuous in the neutral element of S.*

Proof. Let $\theta:L^\infty\to\mathscr{L}^\infty$ be a multiplicative linear lifting. Let Ω denote the Gelfand spectrum of L^∞. θ induces a mapping $\hat{\theta}:[0,1]\to\Omega$ defined by $(\hat{\theta}(x))(f)=(\theta(f))(x)$, $f\in L^\infty$, which immediately is seen to be measurable from the Lebesgue measurable sets in $[0,1]$ to the Baire sets in Ω. Thus the image μ of m under $\hat{\theta}$ is a well defined Radon measure on Ω and from

$$\int_{\hat{S}}\varrho(f)d\mu(\varrho)=\int_0^1(\hat{\theta}(x))(f)dm(x)=\int_0^1\theta(f)dm=\int_0^1 f\,dm=\varphi(f)$$

we see that μ represents φ. Of course we look at Ω as a closed subset of $\hat{S}$; by definition $\mu(\Omega)=1$.

Now let $\omega_0\in\Omega$ and assume $\mu(\{\omega_0\})=0$. Then we can find compact subsets $K_n\subseteqq\Omega-\{\omega_0\}$ with $\mu(K_n)>1-\frac{1}{n}$, $n=1,2,\ldots$. Choose continuous functions $g_n:\Omega\to[0,1]$ such that $g_n|K_n\equiv 1$, but $g_n(\omega_0)=0$. The functions $g_n\circ\hat{\theta}$ belong to S and for $h\in L^1$ we have

$$\left|\int_0^1 h(1-g_n\circ\hat{\theta})dm\right|\leqq\int_{[0,1]\setminus\hat{\theta}^{-1}(K_n)}|h|dm\to 0,\qquad n\to\infty$$

implying $g_n \circ \hat{\theta} \to 1$ in S. But $\omega_0(g_n \circ \hat{\theta}) = g_n(\omega_0) = 0$ for all $n \in \mathbb{N}$ and therefore ω_0 is not continuous at the neutral element of S.

The proof will now be finished by showing that $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$. Let $B \subseteq [0,1]$ be a measurable set. The Gelfand transform of 1_B is the indicator function of a clopen set $A \subseteq \Omega$. From the relation $1_A(\omega) = \omega(1_B)$ we conclude

$$\mu(A) = \int_\Omega 1_A d\mu = \int_0^1 1_A \circ \hat{\theta}(x) dm(x) = \int_0^1 (\hat{\theta}(x))(1_B) dm(x)$$

$$= \int_0^1 (\theta(1_B))(x) dm(x) = \int_0^1 1_B dm = m(B).$$

For a given $\varepsilon > 0$ there exists a partition of $[0,1]$ into finitely many measurable sets, each of which has Lebesgue measure $\leqq \varepsilon$. The corresponding clopen sets of Ω are also a partition of Ω, and this proves that μ is diffuse. □

§ 9. An Approximation Theorem with Relation to a Negative Result by Hilbert

Let p be a polynomial in m variables with real coefficients. p is called positive definite if it is non-negative for all (real) values of the variables. Of course any finite sum of squares of polynomials is positive definite and for $m=1$ and $m=2$ it can be shown that any positive definite polynomial is of this type. However for $m \geqq 3$ this is not true (for the proof of these statements see Hilbert [3]).

Although a positive definite polynomial is not always a finite sum of squares of polynomials, we shall see that it can be approximated by such sums in a reasonable norm. On the vector space P_m of polynomials in m variables with real coefficients we define the norm $|\cdot|_1$ as the sum of the absolute values of the coefficients. If we use the multiindex notation for a polynomial in P_m we obtain a natural identification between P_m and the subspace of $l^1(\mathbb{N}_0^m)$ consisting of functions of finite support. This identification is an isometry and convolution in $l^1(\mathbb{N}_0^m)$ corresponds to product of polynomials.

9.1. Theorem. *Let $p \in P_m$ and suppose that $p(t) \geqq 0$ for all $t \in [-1,1]^m$. Then p can be approximated in $|\cdot|_1$-norm with finite sums of squares of polynomials in P_m.*

Proof. Let C be the $|\cdot|_1$-closure of such sums. Of course C is a convex cone. Suppose that $p \notin C$. By Hahn-Banach's theorem there exists a $|\cdot|_1$-continuous linear functional $\Phi: l^1(\mathbb{N}_0^m) \to \mathbb{R}$ satisfying

(*) $\Phi(f) \geqq 0$ for all $f \in C$

and

$\Phi(p) < 0$.

The functional Φ is induced by a bounded function φ on $\mathbb{N}_0^m$. Condition (*) means in particular that φ is positive definite on $\mathbb{N}_0^m$. Our desintegration shows now $\Phi(p) \geqq 0$ and this contradiction finishes the proof. □

References

0. Bauer, H.: Konvexität in topologischen Vektorräumen. Ausarbeitung einer an der Universität Hamburg gehaltenen Vorlesung (1964)
1. Berg, C., Forst, G.: Potential theory on locally compact abelian groups. Berlin, Heidelberg, New York: Springer 1975
2. Choquet, G.: Theory of capacities. Ann. Inst. Fourier **5**, 131—295 (1954)
3. Hilbert, D.: Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. **32**, 342—350 (1888)
4. Hirsch, F.: Familles résolvantes, générateurs, cogénérateurs, potentiels. Ann. Inst. Fourier **22**, 89—210 (1972)
5. Ressel, P.: Laplace Transformation nichtnegativer und vektorwertiger Masse. Manuscripta math. **13**, 143—152 (1974)
6. Ressel, P.: A short proof of Schoenberg's theorem. Proc. Amer. Math. Soc. **57**, No 1 (1976)
7. Schoenberg, I.J.: Metric spaces and completely monotone functions. Ann. Math. **39**, 811—841 (1938)

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